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Int. J. Heat Mass Transfer. Vol. 23, pp. 726–729
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NATURAL CONVECTION IN A VERTICAL CYLINDRICAL WELL FILLED WITH POROUS MEDIUM

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NOMENCLATURE

a_0, a_2, a_4, a_6, b_0,	coefficients;
c_p,	specific heat at constant pressure;
g,	gravitational acceleration;
h,	thermal conductivity;
k,	permeability;
l,	length of similarity pattern;
L,	depth of well;
Nu,	Nusselt number;
Q,	heat-transfer rate through well opening;
r,	radial position;
R,	well radius;
Ra_L,	Rayleigh number;
T,	temperature;
T_1,	well-wall temperature;
T_2,	reservoir temperature;
$\hat{T}$,	temperature, Oseen solution;
u,	vertical velocity;
$\hat{u}$,	vertical velocity, Oseen solution;
$\hat{u}_\infty$,	vertical velocity infinitely far from wall;
v,	radial velocity;
w,	velocity perpendicular to the wall;
x,	vertical position;
y,	distance from vertical wall;
Y,	total thickness of vertical boundary layer;
$(\ )^*$,	dimensional quality;
$(\ )_c$,	pertaining to the core.

Greek symbols

α,	thermal diffusivity;
β,	coefficient of thermal expansion;
γ,	core radius;
δ,	horizontal length scale, Oseen solution;
μ,	viscosity;
ν,	kinematic viscosity;
ρ,	density;
ψ,	stream function.

1. INTRODUCTION

BOUYANCY-INDUCED convection in fluid-saturated porous media is an important topic in contemporary heat-transfer research. The objective of this article is to outline an analysis of the natural convection mechanism in a vertical cylindrical well filled with porous medium [1]. The well opens into a semi-infinite space filled with the same porous medium. In what follows we analyze the convection pattern generated when the cylindrical wall and the semi-infinite space are maintained at different temperatures.

The present problem is related to the work of Minkowycz and Cheng on convection about a vertical cylinder [2] and about a vertical plane [3]. The Minkówycz and Cheng studies, as well as the present one, are aimed at explaining the interaction between a very large porous reservoir and an irregular impermeable surface bordering the reservoir from above or below. The impermeable surface may protrude into the porous medium, as in [2,3], or it may have concavities filled by the neighboring porous material. The latter set of circumstances is the subject of the present investigation.

2. MATHEMATICAL FORMULATION

We model the fluid-saturated porous medium as homogeneous [4] with the following physical properties: fluid density, ρ; viscosity, μ; coefficient of thermal expansion, β;

thermal conductivity of solid–fluid matrix, k; and thermal diffusivity $\alpha = k/(\rho c_p)$, where c_p is the fluid specific heat at constant pressure. In the cylindrical geometry sketched in Fig. 1 the dimensionless equations governing conservation of mass, momentum and energy in steady-state are

$$\frac{\partial u}{\partial x}+\frac{v}{r}+\frac{\partial v}{\partial r}=0,\quad \frac{\partial u}{\partial r}-\left(\frac{R}{L}\right)^2\frac{\partial v}{\partial x}=\frac{\partial T}{\partial r},\qquad (1,2)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial r}=\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\left(\frac{R}{L}\right)^2\frac{\partial^2 T}{\partial x^2}\qquad (3)$$

where

$$x = x^*/L,\quad r = r^*/R,$$

$$u = u^*R^2/(\alpha L),\quad v = v^*R/\alpha,\qquad (4,5,6,7)$$

$$T=\frac{Kg\beta L(T^*-T_1)}{\alpha\nu}\left(\frac{R}{L}\right)^2.\qquad (8)$$

In definitions (4)–(8) the asterisk indicates the dimensional variables of the problem. In the same definitions, R, L and T_1 are the well radius, height and wall temperature, respectively. The momentum equation (2) is based on the Darcy flow model, while K appearing in equation (8) is the permeability of the porous medium.

It is convenient to place the analysis in the limit in which the well is slender, $R/L \ll 1$. Therefore, equations (2) and (3) will be considered without the terms multiplied by $(R/L)^2$. The appropriate boundary conditions for the solid cylindrical wall are

$$v = 0,\quad T = 0\quad \text{at}\quad r = 1,\qquad (9,10)$$

$$u = 0,\quad T = 0\quad \text{at}\quad x = 0.\qquad (11,12)$$

Special attention must be paid to the boundary condition at $x = 1$ where the well communicates with the semi-infinite reservoir. In Fig. 1 the reservoir is relatively colder ($T_2 < T_1$) and is situated above the well. This is a potentially unstable configuration which leads to fluid motion, cold fluid falling into the well through the middle of the circular cross-section and warmer fluid rising along the heated cylindrical wall. In the analysis we assume that the centerline temperature at the mouth of the well equals the reservoir temperature,

$$T = -Ra_L\left(\frac{R}{L}\right)^2\quad \text{at}\quad x = 1,\quad r = 0,\qquad (13)$$

where Ra_L is the Darcy–Rayleigh number based on the height of the well,

$$Ra_L=\frac{Kg\beta L(T_1-T_2)}{\alpha\nu}>0.\qquad (14)$$

3. THE SIMILARITY REGIME

First, let us consider the existence of a free convection pattern in which the velocity and temperature radial profiles have the same shape independent of vertical position. Upon examining equations (1)–(3) we find that a similarity solution is possible, one with both u and T proportional to x while v is a function only of radial position. The similarity regime is therefore similar to the one found by Lighthill in vertical tubes filled with fluid [5].

Exact analytical solutions for u, v *and* T, given by equations (1)–(3) without the $(R/L)^2$ terms, are impossible due to the nonlinearity associated with the convection terms in equation (3). Following Lighthill's analysis, we seek solutions which satisfy equation (3) in an integral fashion *and* at specified locations, namely, along the centerline $r = 0$ and at the wall $r = 1$:

$$\frac{d}{dx}\left(\int_0^1 ruT\,dr\right)=\left(\frac{\partial T}{\partial r}\right)_{r=1},\qquad (15)$$

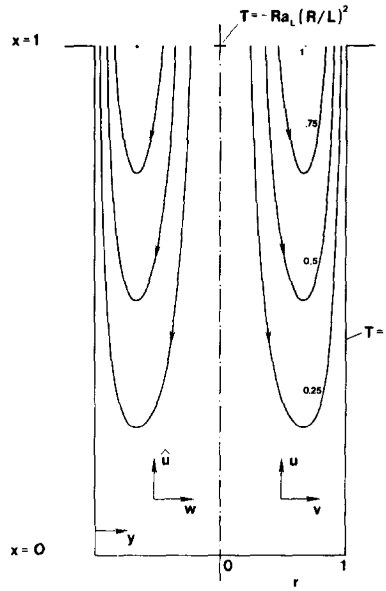

FIG. 1. Streamline pattern in the similarity regime; the numbers on the figure indicate the value of $\psi/(-0.7215)$.

$$\left(u\frac{\partial T}{\partial x}\right)_{r=0}=\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right)_{r=0},$$

$$0=\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right)_{r=1}.\qquad (16,17)$$

Next, we select polynomial expressions for temperature and vertical velocity

$$T = x(a_0 + a_2r^2 + a_4r^4 + a_6r^6),\qquad (18)$$

$$u = x(b_0 + a_2r^2 + a_4r^4 + a_6r^6)\qquad (19)$$

which satisfy the momentum equation (2) identically. The five unknown coefficients appearing in expressions (18) and (19) are determined from conditions (1), (10) and (15)–(17). The result is

$$a_0 = -11.81,\quad a_2 = 21.27,\quad a_4 = -12.781,$$

$$a_6 = 3.317,\quad b_0 = -7.206.\qquad (20)$$

Figure 1 displays a set of streamlines ψ = constant, the streamfunction ψ having been defined in the usual manner by writing $u = (\partial\psi/\partial r)/r$ and $v = -(\partial\psi/\partial x)/r$. Cold fluid creeps down the centerline, gradually warming up the further it reaches into the well. At the same time a layer of warmer fluid rises along the cylindrical wall.

An important feature of the similarity flow is that the depth to which the free convection pattern of Fig. 1 penetrates the well is proportional to the temperature difference driving the flow. More specifically, the temperature condition (13) combined with the temperature distribution (18) yields

$$\frac{l}{L}=0.0847\,Ra_L\left(\frac{R}{L}\right)^2.\qquad (21)$$

Here, l is the physical length of the similarity pattern. In a well of finite depth L, a similarity pattern will exist as long as $l \le L$, i.e. as long as

$$Ra_L\left(\frac{R}{L}\right)^2\le 11.81.\qquad (22)$$

For values of $Ra_L(R/L)^2$ higher than 11.81 the temperature and velocity fields depart from the similarity regime. The critical value 11.81 can only be regarded as approximate, the result of having chosen to replace the energy-conservation statement (3) via conditions (15)–(17). Relying on different sets of conditions, in [1] it is demonstrated that 11.81 is indeed a representative value of $Ra_L(R/L)^2$ above which a similarity regime ceases to exist.

4. THE BOUNDARY LAYER REGIME

We now turn our attention to the high Rayleigh-number limit in which the fluid motion is concentrated in thin boundary layer near the cylindrical wall. We first present an integral solution of the type developed by Lighthill for convection in vertical tubes filled with Newtonian fluid [5]. In the second part of this section we present an alternative solution based on the Oseen linearization technique which recently was shown to give very good results when applied to problems of free convection in enclosures [6–9].

Integral method

Consider the fluid in the well as an isothermal core of radius $\gamma(x)$ and an annular boundary layer of thickness $1 - \gamma(x)$. The piecewise continuous profiles selected for temperature and vertical velocity are

$$T = T_c, \quad u = u_c, \quad 0 < r < \gamma, \tag{23,24}$$

$$T = T_c\left[1 - \left(\frac{r-\gamma}{1-\gamma}\right)^2\right],$$

$$u = u_c - T_c\left(\frac{r-\gamma}{1-\gamma}\right)^2, \quad \gamma < r < 1 \tag{25,26}$$

where T_c and u_c are the core temperature and velocity. The mass conservation condition

$$\int_0^1 ur\,\mathrm{d}r = 0$$

requires

$$u_c = \frac{T_c}{6}(3 - 2\gamma - \gamma^2). \tag{27}$$

Combining equations (23)–(27) with energy integral (15) and recognizing that $T_c = -Ra_L(R/L)^2$ yields

$$\frac{x}{Ra_L(R/L)^2} = \frac{1}{1080}(1 + 18\gamma - 12\gamma^2 - 28\gamma^3 + 15\gamma^4 + 6\gamma^5). \tag{28}$$

Result (28) can be plotted to show that the boundary-layer thickness $1 - \gamma(x)$ grows steadily as the fluid travels upward [1]. This behavior terminates abruptly at $\gamma = 0.392$ which corresponds to a maximum $x/[Ra_L(R/L)^2] = 0.00457 = 1/218.83$. If we set $x = 1$ in equation (28) we obtain the variation of boundary-layer thickness at the mouth, $1 - \gamma(1)$ with Rayleigh number. It is found that $\gamma(1)$ cannot exceed 0.392 which implies that $Ra_L(R/L)^2$ must be greater than 218.83 before the boundary-layer flow (23)–(26) can exist.

Oseen method

Consider a coordinate system $x^* - y^*$, where x^* is measured vertically along the cylindrical wall, as in Fig. 1, and y^* is measured away from the wall ($y^* = R - r^*$). We are interested in the flow and temperature pattern in the annular region close enough to the vertical wall so that $y^* \ll R$. In the cartesian system $x^* - y^*$ the governing equations can be written as

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial w}{\partial y} = 0, \quad \frac{\partial \hat{u}}{\partial y} - \left(\frac{\delta}{L}\right)^2\frac{\partial w}{\partial x} = \frac{\partial \hat{T}}{\partial y}, \tag{29,30}$$

$$\hat{u}\frac{\partial \hat{T}}{\partial x} + w\frac{\partial \hat{T}}{\partial y} = \left(\frac{\delta}{L}\right)^2\frac{\partial^2 \hat{T}}{\partial x^2} + \frac{\partial^2 \hat{T}}{\partial y^2}, \tag{31}$$

with

$$x = x^*/L, \quad y = y^*/\delta, \tag{32,33}$$

$$\hat{u} = u^*\,\delta^2/(\alpha L), \quad w = w^*\,\delta/\alpha,$$

$$\hat{T} = (T - T_1)/(T_1 - T_2). \tag{34,35,36}$$

The horizontal length scale is $\delta = LRa_L^{-1/2}$. Since in the boundary-layer regime $\delta/R \ll 1$, hence $\delta/L \ll 1$, the terms multiplied by $(\delta/L)^2$ in equations (30) and (31) will be neglected. The boundary conditions are

$$w = 0, \quad \hat{T} = 0 \quad \text{at } y = 0, \tag{37,38}$$

$$\hat{u} = \hat{u}_\infty(x), \quad \hat{T} = -1 \quad \text{as } y \to \infty. \tag{39,40}$$

Eliminating the temperature between equations (30) and (31) yields

$$\frac{\partial^2 \hat{u}}{\partial y^2} - (w)\frac{\partial \hat{u}}{\partial y} - (\hat{T}_x)\hat{u} = 0, \tag{41}$$

The Oseen method consists of treating w and $\hat{T}_x$ as unknown functions of x. Thus, equation (41) can be integrated in y subject to conditions (37)–(40)

$$\hat{u} = \exp[yw(x)] + \hat{u}_\infty(x), \quad \hat{T} = \exp[yw(x)] - 1. \tag{42,43}$$

The unknown functions $w(x)$ and $\hat{u}_\infty(x)$ are determined from integral mass and energy conservation statements

$$\int_0^Y \hat{u}\,\mathrm{d}y = 0, \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^Y \hat{u}\hat{T}\,\mathrm{d}y\right) = -\left(\frac{\partial \hat{T}}{\partial y}\right)_{y=0}, \tag{44,45}$$

where Y represents the value of y in the center of the cylinder (from an equal area argument, $Y = R/2\delta$). The final results are [1]

$$w\hat{u}_\infty Y = 1, \quad x = \frac{1}{4w^2} + \frac{2}{3Yw^3}. \tag{46,47}$$

Recognizing that $-1/w$ plays the role of boundary-layer thickness, one can plot this quantity vs x and obtain results qualitatively similar to the boundary-layer thickness produced by the integral method. It is found that the Oseen solution is valid for $Ra_L(R/L)^2 > 768$. The temperature and velocity vary exponentially near the wall, as indicated in equations (42) and (43).

5. HEAT-TRANSFER RESULTS

An important aspect of the free-convection phenomenon is the net heat-transfer interaction between the wall of the vertical cylindrical cavity and the semi-infinite reservoir. In the arrangement shown in Fig. 1 the fluid flow carries heat upward through the mouth of the well at a rate

$$q = 2\pi\int_0^R \rho c_p r^* u^* T^*\,\mathrm{d}r^*, \quad x^* = L. \tag{48}$$

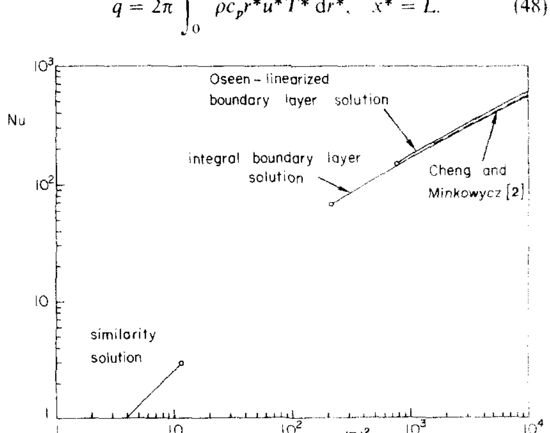

FIG. 2. Summary of heat-transfer results.

The Nusselt number associated with this heat-transfer rate is defined as

$$Nu = \frac{Q}{kL(T_1 - T_2)} = \frac{2\pi}{Ra_L\left(\frac{R}{L}\right)^2}\left(\int_0^1 ruT\,dr\right)_{x=1}. \quad (49)$$

For heat transfer in the similarity regime, equations (18), (19) and (25) were substituted in equation (49) to yield

$$Nu = 0.255\, Ra_L\left(\frac{R}{L}\right)^2. \quad (50)$$

This result appears as a straight line on Fig. 2, in the range $Ra_L(R/L)^2 \leq 11.81$ where the similarity regime exists.

For the boundary-layer regime we developed two solutions. The Nusselt number calculated from the integral solution is shown on Fig. 2 in the range $(R/L)^2 Ra_L \geq 218.83$. Using the calculus of limits, one can show that as $(R/L)^2 Ra_L$ approaches infinity, the Nusselt number is given by

$$Nu = 5.62\frac{R}{L}Ra_L^{1/2}. \quad (51)$$

The Nu result based on the Oseen-linearized boundary-layer solution was also plotted on Fig. 2 in the range $(R/L)^2 Ra_L \geq 768$. In the high Ra_L limit we obtain:

$$Nu = 2\pi\frac{R}{L}Ra_L^{1/2}. \quad (52)$$

Since in the high Rayleigh-number limit the free convection phenomenon inside the well approaches free convection along a flat vertical surface, it is possible to compare results (51) and (52) with the Nusselt number obtained by Cheng and Minkowycz [3] for a vertical plate,

$$Nu = 5.58\frac{R}{L}Ra_L^{1/2}. \quad (53)$$

Comparing asymptotes (51) and (52) with asymptote (53) we find excellent agreement for both boundary-layer solutions developed in this paper. In particular, the integral boundary-layer solution appears to be the better of the two, its asymptotic Nu differing by only 0.7% from the result of Cheng and Minkowycz [3]. This comparison supports the validity of the Karman–Pohlhausen integral method used in this paper to analyze not only the boundary layer regime but also the similarity regime.

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A NOTE ON KUTATELADZE'S EQUATION FOR PARTIAL NUCLEATE BOILING*

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NOMENCLATURE

- h, heat-transfer coefficient in partial nucleate boiling; $h = q/(t_w - t_f)$;
- h_f, one-phase forced-convection heat-transfer coefficient;
- h_b, heat-transfer coefficient during developed boiling [see equation (2) or (3)];
- q, specific heat flux;
- q_b, saturated pool boiling heat-flux corresponding to temperature t_w;
- t_f, bulk temperature;
- t_s, saturation temperature;
- t_w, wall temperature;
- $t_{w,b}$, saturated pool boiling wall temperature corresponding to the heat flux q;
- w, velocity.

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IN THE transition region from forced convection to nucleate boiling, the heat-transfer coefficient is affected by the flow velocity and degree of subcooling. This influence is observed with fluids flowing in tubes or annular spaces and for fluids flowing normally to horizontal cylinders.

In order to evaluate the effect of fluid velocity on the surface boiling heat transfer in tubes at saturation conditions, Kutateladze [1] proposed the following relationship:

$$h/h_f = \left[1 + \left(\frac{h_b}{h_f}\right)^n\right]^{1/n} \quad (1)$$